Correlation functions in isotropic and anisotropic turbulence: the role of the symmetry group

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Abstract

The theory of fully developed turbulence is usually considered in an idealized homogeneous and isotropic state. Real turbulent flows exhibit the effects of anisotropic forcing. The analysis of correlation functions and structure functions in isotropic and anisotropic situations is facilitated and made rational when performed in terms of the irreducible representations of the relevant symmetry group which is the group of all rotations SO(3). In this paper we firstly consider the needed general theory and explain why we expect different (universal) scaling exponents in the different sectors of the symmetry group. We exemplify the theory context of isotropic turbulence (for third order tensorial structure functions) and in weakly anisotropic turbulence (for the second order structure function). The utility of the resulting expressions for the analysis of experimental data is demonstrated in the context of high Reynolds number measurements of turbulence in the atmosphere.

I. INTRODUCTION

Experiments in fluid turbulence are usually limited to the measurement of the velocity field at one single spatial point as a function of time. This situation has begun to improve recently, but still much of the analysis of the statistical properties of Navier-Stokes turbulence

[1] is influenced by this tradition: the Taylor hypothesis is used to justify the identification of velocity signals at different times with differences of longitudinal velocity components across a spatial length scale R. Most of the available statistical information is therefore about properties of longitudinal two-point differences of the Eulerian velocity field and their moments, termed structure functions:

$$S_n(R) = \langle |[\boldsymbol{u}(\boldsymbol{r}+\boldsymbol{R}) - \boldsymbol{u}(\boldsymbol{r})] \cdot \frac{\boldsymbol{R}}{R}|^n \rangle \ , \tag{1}$$

where $\langle \cdot \rangle$ denotes ensemble averaging. In isotropic homogeneous turbulence, these structure functions are expected to behave as a power-law in R, $S_n(R) \sim R^{\zeta_n}$, with apparently universal scaling exponents ζ_n.

Recent research [2,3] has pointed out the advantages of considering not only the longitudinal structure functions, but tensorial multi-point correlations of velocity field differences

$$\boldsymbol{w}(\boldsymbol{r}, \boldsymbol{r}', t) \equiv \boldsymbol{u}(\boldsymbol{r}', t) - \boldsymbol{u}(\boldsymbol{r}, t), \tag{2}$$

given by

$$\begin{aligned} &\hat{\mathcal{F}}_n^{\alpha\beta\ldots\delta}(\boldsymbol{r}_1, \boldsymbol{r}'_1, t_1; \boldsymbol{r}_2, \boldsymbol{r}'_2, t_2; \ldots; \boldsymbol{r}_n, \boldsymbol{r}'_n, t_n) \\ &= \langle w^\alpha(\boldsymbol{r}_1, \boldsymbol{r}'_1, t_1) w^\beta(\boldsymbol{r}_2, \boldsymbol{r}'_2, t_2) \ldots w^\delta(\boldsymbol{r}_n, \boldsymbol{r}'_n, t_n) \rangle \ , \end{aligned} \tag{3}$$

where all the coordinates are distinct. Even when the coordinates fuse to yield time-independent structure functions depending on one separation only, these are tensorial quantities [4] denoted as

$$S^{\alpha\beta\ldots}(\boldsymbol{R}) \equiv \langle [u^\alpha(\boldsymbol{r}+\boldsymbol{R}) - u^\alpha(\boldsymbol{r})][u^\beta(\boldsymbol{r}+\boldsymbol{R}) - u^\beta(\boldsymbol{r})] \ldots \rangle \ . \tag{4}$$

Needless to say, the tensorial information is lost in the usual measurements leading to (1). One of the main points of the present paper is that keeping the tensorial information can help significantly in disentangling different scaling contributions to the statistical objects, contributions that are hard to distinguish when quantities like (1) are considered. Especially when anisotropy implies different tensorial components with possible different scaling exponents characterizing them, careful control of the various contributions is called for.

To understand why irreducible representations of the symmetry group may have an important role in determining the form of correlation functions, we need to discuss the equations of motion which they satisfy. We shall show that the isotropy of the Navier-Stokes equation and the incompressibility condition implies the isotropy of the hierarchical equations which the correlation functions satisfy. We will use this symmetry to show that every component of the general solution with a definite behavior under rotations (i.e., components of a definite *irreducible representation* of the $SO(3)$ group) has to satisfy these equations by itself - independently of components with different behavior under rotations. This "foliation" of the hierarchical equations may possibly lead to different scaling exponents for each component of the general solution which belong to a different $SO(3)$ irreducible representation.

In Sect.2 we describe the general mathematical framework of theory by discussing the structure of tensorial fields from the point of view of $SO(3)$ irreducible representations. We then show in Sect.3 that the hierarchy equations are indeed isotropic and as a result foliate into different sectors of the $SO(3)$ irreducible representations. In the next sections we demonstrate the utility of the theory. In Sect.4 we revisit Kolmogorov's four fifth's law emphasizing the rule of the $SO(3)$ irreducible representations in its derivation. Then, in Sect.5, we present some experimental evidences for the importance of an anisotropic exponent in the second order structure function, in atmospheric measurements. Sect.6 offers conclusions and some comments about the road ahead.

II. TENSORIAL CORRELATION FUNCTIONS AND $SO(3)$ IRREDUCIBLE REPRESENTATIONS: GENERAL THEORY

The physical objects that we deal with are the moments of the velocity field at different space-time locations. In this section we suggest a way of decomposing these objects into components with a definite behavior under rotations. We will show later that components with different behavior under rotation are subject to different dynamical equations, and therefore, possibly, scale differently. Essentially, we are about to describe the tenso-

rial generalization of the well-known procedure of decomposing a scalar function $\Psi(\mathbf{r})$ into components of different irreducible representations using the spherical harmonics:

$$\Psi(\mathbf{r}) = \sum_{l,m} a_{lm}(r) Y_{lm}(\hat{\mathbf{r}}) \ . \tag{5}$$

A. Formal definition

Consider a typical moment of the velocity field, Eq.(3). $F_n^{\alpha_1 \dots \alpha_n}(\mathbf{r}_1, \mathbf{r}'_1, t_1; \dots ; \mathbf{r}_n, \mathbf{r}'_n, t_n)$ is a function of $2n$ spatial variables and n temporal variables. Physically, it is a *tensor field*: if $\mathbf{F}_n$ is measured in two frames I and $\overline{I}$ which are connected by the spatial transformation (say, a rotation)

$$\overline{x}^\alpha = \Lambda^\alpha{}_\beta x^\beta \tag{6}$$

then, the measured quantities in each frame will be connected by the relation:

$$\begin{aligned} & \overline{F}_n^{\alpha_1 \dots \alpha_n}(\overline{\mathbf{r}}_1, \overline{\mathbf{r}}'_1, \overline{t}_1; \dots ; \overline{\mathbf{r}}_n, \overline{\mathbf{r}}'_n, \overline{t}_n) \\ &= \Lambda^{\alpha_1}{}_{\beta_1} \cdots \Lambda^{\alpha_n}{}_{\beta_n} F_n^{\beta_1 \dots \beta_n}(\mathbf{r}_1, \mathbf{r}'_1, t_1; \dots ; \mathbf{r}_n, \mathbf{r}'_n, t_n) \\ &= \Lambda^{\alpha_1}{}_{\beta_1} \cdots \Lambda^{\alpha_n}{}_{\beta_n} F_n^{\beta_1 \dots \beta_n}(\Lambda^{-1}\overline{\mathbf{r}}_1, \Lambda^{-1}\overline{\mathbf{r}}'_1, \overline{t}_1; \dots ; \Lambda^{-1}\overline{\mathbf{r}}_n, \Lambda^{-1}\overline{\mathbf{r}}'_n, \overline{t}_n) \ . \end{aligned} \tag{7}$$

We see that as we move from one frame to another, the *functional form* of the tensor field changes. We want to classify the different tensor fields according to the change in their functional form as we make that move. We can omit the time variables from our discussion since under rotation they merely serve as parameters.

Consider coordinate transformations which are pure rotations. For such transformations we may simplify the discussion by separating the dependence on the amplitude of $\mathbf{r}_i$ from the dependence on the directionality of $\mathbf{r}_i$:

$$\begin{aligned} & T^{\alpha_1 \dots \alpha_n}(\mathbf{r}_1, \dots, \mathbf{r}_p) \\ &= T^{\alpha_1 \dots \alpha_n}(r_1, \dots, r_p; \hat{\mathbf{r}}_1, \dots, \hat{\mathbf{r}}_p) \end{aligned}$$

For pure rotations we may treat the amplitudes $r_1, \ldots, r_p$ as parameters: the transformations properties of $T^{\alpha_1 \ldots \alpha_n}$ under rotation are determined only by the dependence of $T^{\alpha_1 \ldots \alpha_n}$ on the unit vectors $\hat{\mathbf{r}}_1, \ldots, \hat{\mathbf{r}}_p$. Accordingly it seems worthwhile to discuss tensor fields which are functions of the unit vectors *only*. Notice that in the scalar case we follow the same procedure: by restricting our attention to scalar functions that depend only on the unit vector $\hat{\mathbf{r}}$, we construct the spherical harmonics. These functions are *defined* such that each one of them has unique transformation properties under rotations. We then represent the most general scalar function as a linear combination of the spherical harmonics with r-dependent coefficients, see Eq. (5).

The classification of the tensor fields $T^{\alpha_1 \ldots \alpha_n}(\hat{\mathbf{r}}_1, \ldots, \hat{\mathbf{r}}_p)$ according to their functional change under rotations follows immediately from group representation theory [5,6]. But in order to demonstrate that, we must first make some formal definitions. We define $\mathcal{S}_p^n$ to be the space of all smooth tensor fields of rank n which depend on p unit vectors. This is obviously a linear space of infinite dimension. With each rotation $\Lambda \in SO(3)$, we may now associate a linear transformation $\mathcal{O}_\Lambda$ on that space via the relation (7):

$$
\begin{aligned}
&[\mathcal{O}_\Lambda T]^{\alpha_1, \ldots, \alpha_n}(\hat{\mathbf{r}}_1, \ldots, \hat{\mathbf{r}}_p) \\
&\equiv \Lambda^{\alpha_1}{}_{\beta_1} \cdots \Lambda^{\alpha_n}{}_{\beta_n} T^{\beta_1 \ldots \beta_n}(\Lambda^{-1}\hat{\mathbf{r}}_1, \ldots, \Lambda^{-1}\hat{\mathbf{r}}_p).
\end{aligned}
$$

Using this definition, it is easy to see that the set of linear operators $\mathcal{O}_\Lambda$ furnishes a representation of the rotation group $SO(3)$ since they satisfy the relations:

$$
\begin{aligned}
\mathcal{O}_{\Lambda_1}\mathcal{O}_{\Lambda_2} &= \mathcal{O}_{\Lambda_1\Lambda_2} \\
\mathcal{O}_\Lambda^{-1} &= \mathcal{O}_{\Lambda^{-1}}.
\end{aligned}
$$

General group theoretical considerations imply that it is possible to decompose $\mathcal{S}_p^n$ into subspaces which are invariant to the action of all the group operators $\mathcal{O}_\Lambda$. Moreover, we can choose these subspaces to be *irreducible* in the sense that they will not contain any invariant subspace themselves (excluding themselves and the trivial subspace of the zero tensor field). For the $SO(3)$ group each of these subspaces is conventionally characterized by an integer

$j = 0, 1, 2, \ldots$ and is of dimension $2j+1$ [5,6]. It should be noted that unlike the scalar case, in the general space $\mathcal{S}_p^n$, there might be more than one subspace for each given value of j. We therefore use the index q to distinguish subspaces with the same j. For each irreducible subspace (q, j) we can now choose a basis with $2j+1$ components labeled by the index m:

$$B_{qjm}^{\alpha_1,\ldots,\alpha_n}(\hat{\mathbf{r}}_1, \ldots, \hat{\mathbf{r}}_p) \ ; \ m = -j, \ldots + j.$$

In each subspace (q, j), the group operators $\mathcal{O}_\Lambda$ furnish a $2j+1$ dimensional irreducible representation of $SO(3)$. Using the basis $B_{qjm}^{\alpha_1,\ldots,\alpha_n}(\hat{\mathbf{r}}_1, \ldots, \hat{\mathbf{r}}_p)$, we can represent each operator $\mathcal{O}_\Lambda$ as a $(2j+1) \times (2j+1)$ matrix $D_{m'm}^{(j)}(\Lambda)$ via the relation:

$$\begin{aligned}
&[\mathcal{O}_\Lambda B]_{qjm}^{\alpha_1,\ldots,\alpha_n}(\hat{\mathbf{r}}_1, \ldots, \hat{\mathbf{r}}_p) \\
&= \Lambda^{\alpha_1}{}_{\beta_1} \cdots \Lambda^{\alpha_n}{}_{\beta_n} B_{qjm}^{\beta_1\ldots\beta_n}(\Lambda^{-1}\hat{\mathbf{r}}_1, \ldots, \Lambda^{-1}\hat{\mathbf{r}}_p) \\
&\equiv \sum_{m'=-j}^{+j} D_{m'm}^{(j)}(\Lambda) B_{qjm'}^{\alpha_1,\ldots,\alpha_n}(\hat{\mathbf{r}}_1, \ldots, \hat{\mathbf{r}}_p).
\end{aligned}$$

It is conventional to choose the basis $\mathbf{B}_{qjm}$ such that the matrices $D_{m'm}^{(j)}(\phi)$, that correspond to rotations of ϕ radians around the 3 axis, will be diagonal, and given by:

$$D_{m'm}^{(j)}(\phi) = \delta_{mm'} e^{im\phi}.$$

The $\mathcal{S}_p^n$ space possesses a very natural inner-product:

$$\begin{aligned}
\langle \mathbf{T}, \mathbf{U} \rangle \equiv &\int d\hat{\mathbf{x}}_1 \ldots d\hat{\mathbf{x}}_p \\
&\cdot T^{\alpha_1\ldots\alpha_n}(\hat{\mathbf{x}}_1 \ldots \hat{\mathbf{x}}_p) g_{\alpha_1\beta_1} \ldots g_{\alpha_n\beta_n} U^{\beta_1\ldots\beta_n}(\hat{\mathbf{x}}_1 \ldots \hat{\mathbf{x}}_p)^*
\end{aligned}$$

where $g_{\alpha\beta}$ is the 3-dimensional Euclidean metric tensor:

$$g_{\alpha\beta} = \begin{pmatrix} 1 & & \\ & 1 & \\ & & 1 \end{pmatrix}.$$

By definition, the rotation matrices $\Lambda^\alpha{}_\beta$ preserve this metric, and therefore it is easy to see that for each $\Lambda \in SO(3)$ we get:

$$\langle \mathcal{O}_\Lambda \mathbf{T}, \mathcal{O}_\Lambda \mathbf{U} \rangle = \langle \mathbf{T}, \mathbf{U} \rangle$$

so that, $\mathcal{O}_\Lambda$ are unitary operators. If we now choose the basis $\mathbf{B}_{qjm}$ to be orthonormal with respect to the inner-product defined above, then the matrices $D^{(j)}_{m'm}(\Lambda)$ will be unitary.

Finally, consider *isotropic tensor fields*. An isotropic tensor field is a tensor field which preserves its functional form under any arbitrary rotation of the coordinate system. In other words, it is a tensor field which is invariant to the action of all operators $\mathcal{O}_\Lambda$. The one dimensional subspace spanned by this tensor-field is therefore invariant under all operators $\mathcal{O}_\Lambda$ and therefore it must be a $j = 0$ subspace.

Once the basis $\mathbf{B}_{qjm}$ has been selected, we may expand any arbitrary tensor field $F^{\alpha_1 \dots \alpha_n}(\mathbf{r}_1, \dots, \mathbf{r}_p)$ in this basis. As mentioned above, for each fixed set of amplitudes $r_1, \dots, r_p$, we can regard the tensor field $F^{\alpha_1 \dots \alpha_n}(\mathbf{r}_1, \dots, \mathbf{r}_p)$ as a tensor field which depends only on the unit vectors $\hat{\mathbf{r}}_1, \dots, \hat{\mathbf{r}}_p$, and hence belongs to $\mathcal{S}^n_p$. We can therefore expand it in terms of the basis tensor fields $\mathbf{B}_{qjm}$ with coefficients that depend on $r_1, \dots, r_p$:

$$\begin{aligned} & F^{\alpha_1 \dots \alpha_n}(\mathbf{r}_1, \dots, \mathbf{r}_p) \\ &= \sum_{q,j,m} a_{qjm}(r_1, \dots, r_p) B^{\alpha_1, \dots, \alpha_n}_{qjm}(\hat{\mathbf{r}}_1, \dots, \hat{\mathbf{r}}_p) \ . \end{aligned} \tag{8}$$

The goal of the following sections is to demonstrate the utility of such expansions for the study of the scaling properties of the correlation functions.

B. Construction of the basis tensors

a. The Clebsch-Gordan machinery .

A straightforward (although somewhat impractical) way to construct the basis tensors $\mathbf{B}_{qjm}$ is to use the well-known Clebsch-Gordan machinery. In this approach we consider the $\mathcal{S}^n_p$ space as a *direct product space* of n 3-dimensional Euclidean vector spaces with p infinite dimensional spaces of single-variable continuous functions on the unit sphere. In other words, we notice that $\mathcal{S}^n_p$ is given by:

$$\mathcal{S}^n_p = \left[\mathcal{S}^1_0\right]^n \otimes \left[\mathcal{S}^0_1\right]^p ,$$

and therefore every tensor $T^{\alpha_1\ldots\alpha_n}(\hat{\mathbf{r}}_1\ldots\hat{\mathbf{r}}_p)$ can be represented as a linear combination of tensors of the form:

$$v_1^{\alpha_1}\ldots v_n^{\alpha_n}\varphi_1(\hat{\mathbf{r}}_1)\cdot\ldots\cdot\varphi_p(\hat{\mathbf{r}}_p)\,.$$

$v_i^{\alpha_i}$ are constant Euclidean vectors and $\varphi_i(\hat{\mathbf{r}}_i)$ are continuous functions over the unit sphere. The 3-dimensional Euclidean vector space, $\mathcal{S}_0^1$, contains exactly one irreducible representation of $SO(3)$ - the $j=1$ representation - while $\mathcal{S}_1^0$, the space of continuous functions over the unit sphere, contains every irreducible representation exactly once. The statement that $\mathcal{S}_p^n$ is a direct product space may now be written in a group representation notation as:

$$\mathcal{S}_p^n = \overbrace{1\otimes 1\otimes\ldots\otimes 1}^{n\ \text{times}}\otimes\overbrace{(0\oplus 1\oplus 2\ldots)\otimes\ldots(0\oplus 1\oplus 2\ldots)}^{p\ \text{times}}$$

We can now choose an appropriate basis for each space in the product:

- For the 3-dimensional Euclidean space we may choose:

$$\mathbf{e}_1=\frac{1}{\sqrt{2}}\begin{pmatrix}1\\ i\\ 0\end{pmatrix},\quad \mathbf{e}_0=\frac{1}{\sqrt{2}}\begin{pmatrix}0\\ 0\\ 1\end{pmatrix},\quad \mathbf{e}_{-1}=\frac{1}{\sqrt{2}}\begin{pmatrix}1\\ -i\\ 0\end{pmatrix}$$

- For the space of continuous functions over the unit sphere we may choose the well-known spherical harmonic functions.

Once these bases have been chosen, we can construct a direct-product basis for $\mathcal{S}_p^n$:

$$\begin{aligned}&E^{\alpha_1\ldots\alpha_n}_{i_1\ldots i_n(l_1\mu_1)\ldots(l_p\mu_p)}(\hat{\mathbf{r}}_1,\ldots,\hat{\mathbf{r}}_p)\\ &\equiv e^{\alpha_1}_{i_1}\cdot\ldots\cdot e^{\alpha_n}_{i_n}\cdot Y_{l_1,\mu_1}(\hat{\mathbf{r}}_1)\cdot\ldots\cdot Y_{l_p,\mu_p}(\hat{\mathbf{r}}_p)\end{aligned}$$

The unitary matrix that connects the $\mathbf{E}_{i_1\ldots i_n(l_1\mu_1)\ldots(l_p\mu_p)}$ basis to the $\mathbf{B}_{qjm}$ basis can be calculated using the appropriate Clebsch-Gordan coefficients. The calculation is straightforward but very long and tedious. However, the above analysis enables us to count and classify the

different irreducible representations of a given j. By using the standard rules of angular-momentum addition:

$$s \otimes l = |s - l| \oplus \ldots \oplus (s + l)$$

we can count the number of irreducible representations of a given j. For example, consider the space $\mathcal{S}_1^2$ of second-rank tensors with one variable over the unit sphere. Using the angular-momentum addition rules we see:

$$\begin{aligned}\mathcal{S}_1^2 &= 1 \otimes 1 \otimes (0 \oplus 1 \oplus 2 \oplus 3 \oplus \ldots) \\ &= (0 \oplus 1 \oplus 2) \otimes (0 \oplus 1 \oplus 2 \oplus 3 \oplus \ldots) \\ &= (3 \times 0) \oplus (7 \times 1) \oplus (9 \times 2) \oplus (9 \times 3) \oplus \ldots\end{aligned} \tag{9}$$

We see that there are exactly three $j = 0$ representations, seven $j = 1$ representations and 9 representations for each $j > 1$. It can be further argued that the symmetry properties of the basis tensors with respect to their indices come from the $1 \otimes 1 = 0 \oplus 1 \oplus 2$ part of the direct product (9). Therefore, out of the 9 irreducible representation of $j > 1$, 5 will be symmetric and traceless, 3 will be anti-symmetric and 1 will be tracefull and diagonal. Similarly, the parity of the resulting tensors (with respect to the single variable) can be calculated.

Once we know how many irreducible representations of each j are found in $\mathcal{S}_p^n$, we can construct them "by hand", in some other, more practical method which will be demonstrated in the following subsection.

b. Alternative derivation of the $\mathbf{B}_{qjm}$.

The method we wish to propose in this subsection is based on the simple idea that contractions with $r^\alpha, \delta^{\alpha\beta}, \epsilon^{\alpha\beta\gamma}$ and differentiation with respect to r^α are all *isotropic* operations. Isotropic in the sense that the resulting expression will have the *same* transformation properties under rotation as the expression we started with. The proof of the last statement follows directly from the transformation properties of $r^\alpha, \delta^{\alpha\beta}, \epsilon^{\alpha\beta\gamma}$.

The construction of all $\mathbf{B}_{qjm}$ that belongs to $\mathcal{S}_1^n$ now becomes a rather trivial task. We begin by defining a scalar tensor field with a definite j, m. An obvious choice will be the

well-known spherical harmonics $Y_{jm}(\hat{\mathbf{r}})$, but a better one will be:

$$\Phi_{jm}(\mathbf{r}) \equiv r^j Y_{jm}(\hat{\mathbf{r}}).$$

The reason that we prefer $\Phi_{jm}(\mathbf{r})$ to $Y_{jm}(\hat{\mathbf{r}})$, is that $\Phi_{jm}(\mathbf{r})$ is polynomial in $\mathbf{r}$ (while $Y_{jm}(\hat{\mathbf{r}})$ is polynomial in $\hat{\mathbf{r}}$) and therefore it is easier to differentiate it with respect to $\mathbf{r}$. Once we have defined $\Phi_{jm}(\mathbf{r})$, we can construct the $\mathbf{B}_{qjm}$ by "adding indices" to $\Phi_{jm}(\mathbf{r})$ using the isotropic operations mentioned above. For example, we may now construct:

- $r^{-j}\delta^{\alpha\beta}\Phi_{jm}(\mathbf{r})$,
- $r^{-j+2}\delta^{\alpha\beta}\partial^{\tau}\partial^{\gamma}\Phi_{jm}(\mathbf{r})$,
- $r^{-j-1}x^{\alpha}\Phi_{jm}(\mathbf{r})$, etc...

Notice that we should always multiply the resulting expression with an appropriate power of r, in order to make it r-independent, and thus a function of $\hat{\mathbf{r}}$ only.

The crucial role of the Clebsch-Gordan analysis is to tell us how many representations from each type we should come up with. First, it tells us the highest power of $\hat{\mathbf{r}}$ in each representation, and then it can also give us the symmetry properties of $\mathbf{B}_{qjm}$ with respect to their indices. For example, consider the irreducible representations of $\mathcal{S}_1^2$ - second rank tensors which depend on one unit vector $\hat{\mathbf{r}}$. The Clebsch-Gordan analysis shows us that this space contains the following irreducible representations:

$$\begin{aligned}\mathcal{S}_1^2 &= \mathcal{S}_0^1 \otimes \mathcal{S}_0^1 \otimes \mathcal{S}_1^0 \\ &= 1 \otimes 1 \otimes (0 \oplus 1 \oplus 2 \oplus 3 \oplus \ldots) \\ &= (0 \oplus 1 \oplus 2) \otimes (0 \oplus 1 \oplus 2 \oplus 3 \oplus \ldots) \\ &= (3 \times 0) \oplus (7 \times 1) \oplus (9 \times 2) \oplus (9 \times 3) \oplus \ldots\end{aligned}$$

That is, for each $j > 1$ we're going to have 9 irreducible representations. The indices symmetry of the tensor comes from the $\mathcal{S}_0^1 \otimes \mathcal{S}_0^1 = 1 \otimes 1 = 0 \oplus 1 \oplus 2$ part of the direct product. This is a direct product of two Euclidean spaces, so its a second rank constant

tensor. We can mark the representation number in this space with the letter s, and the representation number of the $\mathcal{S}_1^0 = 0 \oplus 1 \oplus 2 \oplus 3 \oplus \ldots$ space with the letter l. This way each representation in $\mathcal{S}_1^2$ of a given j will have two additional numbers (s, l), which actually serve as the index q that distinguishes irreducible representations of the same j. The s index will determine the indices symmetry of the tensor, while the l index will determine the highest power of $\hat{\mathbf{r}}$ in the tensor. If we now recall that in the space of constant second-rank tensors, $\mathcal{S}_0^1 \otimes \mathcal{S}_0^1 = 0 \oplus 1 \oplus 2$, the $s = 0, 2$ representations are symmetric while the $s = 1$ representation is anti-symmetric, we can easily construct the $\mathbf{B}_{qjm}$:

$$
\begin{aligned}
&(s,l) = (0, j) && B_{1jm}(\hat{\mathbf{r}}) \equiv r^{-j} \delta^{\alpha\beta} \Phi_{jm}(\mathbf{r}), \\
&(s,l) = (1, j-1) && B_{2jm}(\hat{\mathbf{r}}) \equiv r^{-j+1} \epsilon^{\alpha\beta\mu} \partial_\mu \Phi_{jm}(\mathbf{r}), \\
&(s,l) = (1, j) && B_{3jm}(\hat{\mathbf{r}}) \equiv r^{-j} \left[r^\alpha \partial^\beta - r^\beta \partial^\alpha \right] \Phi_{jm}(\mathbf{r}), \\
&(s,l) = (1, j+1) && B_{4jm}(\hat{\mathbf{r}}) \equiv r^{-j-1} \epsilon^{\alpha\beta\mu} r_\mu \Phi_{jm}(\mathbf{r}), \\
&(s,l) = (2, j-2) && B_{5jm}(\hat{\mathbf{r}}) \equiv r^{-j+2} \partial^\alpha \partial^\beta \Phi_{jm}(\mathbf{r}), \\
&(s,l) = (2, j-1) && B_{6jm}(\hat{\mathbf{r}}) \equiv r^{-j+1} \left[\epsilon^{\alpha\mu\nu} r_\mu \partial_\nu \partial^\beta + \epsilon^{\beta\mu\nu} r_\mu \partial_\nu \partial^\alpha \right] \Phi_{jm}(\mathbf{r}), \\
&(s,l) = (2, j) && B_{7jm}(\hat{\mathbf{r}}) \equiv r^{-j} \left[r^\alpha \partial^\beta + r^\beta \partial^\alpha \right] \Phi_{jm}(\mathbf{r}), \\
&(s,l) = (2, j+1) && B_{8jm}(\hat{\mathbf{r}}) \equiv r^{-j-1} \left[r^\beta \epsilon^{\alpha\mu\nu} r_\mu \partial_\nu + r^\alpha \epsilon^{\beta\mu\nu} r_\mu \partial_\nu \right] \Phi_{jm}(\mathbf{r}), \\
&(s,l) = (2, j+2) && B_{9jm}(\hat{\mathbf{r}}) \equiv r^{-j-2} r^\alpha r^\beta \Phi_{jm}(\mathbf{r}).
\end{aligned}
\tag{10}
$$

It should be stressed that these $\mathbf{B}_{qjm}$ are not exactly the same one we would have gotten from the Clebsch-Gordan machinery. For example, they are not orthogonal among themselves for the same values of j, m (although, they are orthogonal for different values of j or m). Nevertheless, they are linearly independent and thus span a given (j, m) sector in the $\mathcal{S}_1^2$ space.

III. THE ISOTROPY OF THE HIERARCHY OF EQUATIONS AND ITS CONSEQUENCES

In this section we derive the equations of motion for the statistical averages of the velocity and pressure fields differences. We start from the Navier-Stokes equations, and show that

its isotropy implies the isotropy of the equations for the statistical objects. Finally, we demonstrate the foliation of these equations to different sectors of j, m.

Consider a Navier-Stokes incompressible turbulence in a bounded domain Ω. The equations of motion describing the flow are:

$$\partial_t u^\alpha + u^\mu \partial_\mu u^\alpha = -\partial^\alpha p + \nu \partial^2 u^\alpha \ ,$$
$$\partial_\alpha u^\alpha = 0 \ .$$

As is well known, the relevant dynamical time scales are revealed only when the effect of sweeping is removed. In our work we use the Belinicher-L'vov transformation [7] in which the flow is observed from the point of view of one specific fluid particle which is located at $\mathbf{r}_0$ at time t_0. Let $\rho(\mathbf{r}_0, t_0|t)$ be the particle's translation at time t:

$$\rho(\mathbf{r}_0, t_0|t) = \int_{t_0}^{t} ds \mathbf{u}[\mathbf{r}_0 + \rho(\mathbf{r}_0, t_0|s), s] \ . \tag{11}$$

We then redefine the velocity and pressure fields to be those seen from an inertial frame whose origin sits at the current particle's position:

$$\mathbf{v}(\mathbf{r}_0, t_0|\mathbf{r}, t) \equiv \mathbf{u}[\mathbf{r} + \rho(\mathbf{r}_0, t_0|t), t] \ ,$$
$$\pi(\mathbf{r}_0, t_0|\mathbf{r}, t) \equiv p[\mathbf{r} + \rho(\mathbf{r}_0, t_0|t), t] \ .$$

Next, we define the differences of these fields:

$$\mathcal{W}^\alpha(\mathbf{r}_0, t_0|\mathbf{r}, \mathbf{r}', t) \equiv v^\alpha(\mathbf{r}_0, t_0|\mathbf{r}, t) - v^\alpha(\mathbf{r}_0, t_0|\mathbf{r}', t) \ ,$$
$$\Pi(\mathbf{r}_0, t_0|\mathbf{r}, \mathbf{r}', t) \equiv \pi(\mathbf{r}_0, t_0|\mathbf{r}, t) - \pi(\mathbf{r}_0, t_0|\mathbf{r}', t) \ .$$

A straightforward calculation shows that the dynamical equations for $\mathcal{W}$ are:

$$\begin{aligned}\partial_t \mathcal{W}^\alpha(\mathbf{r}, \mathbf{r}', t) = &- \left(\partial^\alpha + \partial'^\alpha\right) \Pi(\mathbf{r}_0, t_0|\mathbf{r}, \mathbf{r}', t) \\ &+\nu \left(\partial^2 + \partial'^2\right) \mathcal{W}^\alpha(\mathbf{r}_0, t_0|\mathbf{r}, \mathbf{r}', t) \\ &-\partial_\mu \mathcal{W}^\mu(\mathbf{r}_0, t_0|\mathbf{r}, \mathbf{r}_0, t)\mathcal{W}^\alpha(\mathbf{r}_0, t_0|\mathbf{r}, \mathbf{r}', t) \\ &-\partial'_\mu \mathcal{W}^\mu(\mathbf{r}_0, t_0|\mathbf{r}', \mathbf{r}_0, t)\mathcal{W}^\alpha(\mathbf{r}_0, t_0|\mathbf{r}, \mathbf{r}', t) \ ,\end{aligned} \tag{12}$$
$$\partial_\alpha \mathcal{W}^\alpha(\mathbf{r}_0, t_0|\mathbf{r}, \mathbf{r}', t) = \partial'_\alpha \mathcal{W}^\alpha(\mathbf{r}_0, t_0|\mathbf{r}, \mathbf{r}', t) = 0 \ .$$

By inspection, t_0 merely serves as a parameter, and therefore we will not denote it explicitly in the following discussion. Also, in order to make the equations easier to understand, let us introduce some shorthand notation for the variables $(\mathbf{r}_k, \mathbf{r}'_k, t_k)$:

$$\mathbf{X}_k \equiv (\mathbf{r}_k, \mathbf{r}'_k, t_k) \ ,$$

$$X_k \equiv (r_k, r'_k, t_k) \ ,$$

$$\hat{\mathbf{X}}_k \equiv (\hat{\mathbf{r}}_k, \hat{\mathbf{r}}'_k) \ .$$

Using (12), we can now derive the dynamical equations for the statistical moments of $\mathcal{W}$,Π: Let $\langle\cdot\rangle$ denote a suitable ensemble averaging. We define two types of statistical moments:

$$\begin{aligned} &\mathcal{F}_n^{\alpha_1\ldots\alpha_n}(\mathbf{r}_0|\mathbf{X}_1,\ldots,\mathbf{X}_n) \\ \equiv\ &\langle \mathcal{W}^{\alpha_1}(\mathbf{r}_0|\mathbf{X}_1)\ldots\mathcal{W}^{\alpha_n}(\mathbf{r}_0|\mathbf{X}_n)\rangle \ , \\ &\mathcal{H}_n^{\alpha_2\ldots\alpha_n}(\mathbf{r}_0|\mathbf{X}_1,\ldots,\mathbf{X}_n) \\ \equiv\ &\langle \Pi(r_0|\mathbf{X}_1)\mathcal{W}^{\alpha_2}(\mathbf{r}_0|\mathbf{X}_2)\ldots\mathcal{W}^{\alpha_n}(\mathbf{r}_0|\mathbf{X}_n)\rangle \ . \end{aligned}$$

Equation (12) implies:

$$\begin{aligned} &\partial_{t_1}\mathcal{F}_n^{\alpha_1\ldots\alpha_n}(\mathbf{r}_0|\mathbf{X}_1,\ldots,\mathbf{X}_n) \\ =\ &-\left(\partial^{\alpha_1}_{(r_1)} + \partial^{\alpha_1}_{(r'_1)}\right)\mathcal{H}_n^{\alpha_2\ldots\alpha_n}(\mathbf{r}_0|\mathbf{X}_1,\ldots,\mathbf{X}_n) \\ &-\partial_\mu^{(r_1)}\mathcal{F}_{n+1}^{\mu\alpha_1\ldots\alpha_n}\left(\mathbf{r}_0|\tilde{\mathbf{X}}, X_1,\ldots,\mathbf{X}_n\right) \\ &-\partial_\mu^{(r'_1)}\mathcal{F}_{n+1}^{\mu\alpha_1\ldots\alpha_n}\left(\mathbf{r}_0|\tilde{\mathbf{X}}', \mathbf{X}_1,\ldots,\mathbf{X}_n\right) \\ &+\nu\left(\partial^2_{(r_1)} + \partial^2_{(r'_1)}\right)\mathcal{F}_n^{\alpha_1\ldots\alpha_n}(\mathbf{r}_0|\mathbf{X}_1,\ldots,\mathbf{X}_n) \ , \end{aligned} \tag{13}$$

$$\tilde{\mathbf{X}} \equiv (\mathbf{r}_0, \mathbf{r}', t) \ ; \ \tilde{\mathbf{X}}' \equiv (\mathbf{r}, \mathbf{r}_0, t) \ , \tag{14}$$

$$\begin{aligned} &\partial_{\alpha_1}^{(r_1)}\mathcal{F}_n^{\alpha_1\ldots\alpha_n}(\mathbf{r}_0|\mathbf{X}_1,\ldots,\mathbf{X}_n) = 0 \ , \\ &\partial_{\alpha_1}^{(r'_1)}\mathcal{F}_n^{\alpha_1\ldots\alpha_n}(\mathbf{r}_0|\mathbf{X}_1,\ldots,\mathbf{X}_n) = 0 \ . \end{aligned} \tag{15}$$

Equations (13), (15) are linear and homogeneous. Therefore their solutions form a linear space. The most general solution to these equations is given by a linear combination of

a suitable basis of the solutions space. To construct a specific solution, we must use the boundary conditions in order to set the linear weights of the basis solutions. We shall now show that the isotropy of these equations implies that our basis of solutions can be constructed such that every solution will have a definite behavior under rotations (that is, definite j and m - see Sect. 2). But before we do that, note that in many aspects the situation described here is similar to the well-known problem of Laplace equation in a closed domain Ω:

$$\nabla^2\Psi = 0 \ ,$$

$$\Psi|_{\partial\Omega} = \sigma \ .$$

The Laplace equation is linear, homogeneous and isotropic. Therefore its solutions form a linear space. One possible basis for this space is:

$$\Psi_{l,m}(\mathbf{r}) \equiv r^l Y_{lm}(\hat{\mathbf{r}}) \ ,$$

in which the solutions have a definite behavior under rotations (belong to an irreducible representation of $SO(3)$). The general solution of the problem is given as a linear combination of the $\Psi_{l,m}(\mathbf{r})$:

$$\Psi(\mathbf{r}) = \sum_{l,m} a_{l,m}\Psi_{l,m}(\mathbf{r}) \ .$$

For a specific problem, we use the value of $\Psi(\mathbf{r})$ on the boundary (i.e., we use $\sigma(\mathbf{r})$) in order to set the values of $a_{l,m}$.

To see that the same thing happens with the hierarchy equations (13, 15), we consider an arbitrary solution $\{\boldsymbol{\mathcal{F}}_n, \boldsymbol{\mathcal{H}}_n|\ n = 2,3,\ldots\}$ of these equations. According to Sect.2 we may write the tensor fields $\boldsymbol{\mathcal{F}}_n, \boldsymbol{\mathcal{H}}_n$ in terms of a basis $\mathbf{B}_{qjm}$:

$$\begin{aligned} &\mathcal{F}_n^{\alpha_1\ldots\alpha_n}(\mathbf{r}_0|\mathbf{X}_1,\ldots,\mathbf{X}_n) \\ \equiv &\sum_{q,j,m} F^{(n)}_{qjm}(r_0, X_1,\ldots,X_n) \\ &\times B^{(n)\ \alpha_1\ldots\alpha_n}_{qjm}(\hat{\mathbf{r}}_0, \hat{\mathbf{X}}_1,\ldots\hat{\mathbf{X}}_n) \ , \end{aligned} \tag{16}$$

$$
\begin{aligned}
&\mathcal{H}_n^{\alpha_2\ldots\alpha_n}(\mathbf{r}_0|\mathbf{X}_1,\ldots,\mathbf{X}_n)\\
\equiv &\sum_{q,j,m} H^{(n)}_{qjm}(r_0,X_1,\ldots,X_n)\\
&\times B^{(n-1)\ \alpha_2\ldots\alpha_n}_{qjm}(\hat{\mathbf{r}}_0,\hat{\mathbf{X}}_1,\ldots\hat{\mathbf{X}}_n)\ .
\end{aligned}
$$

Now all we have to show is that the pieces of $\boldsymbol{\mathcal{F}}_n,\boldsymbol{\mathcal{H}}_n$ with definite j,m solve the hierarchy equations *by themselves* - independently of pieces with different j,m. The proof of the last statement is straightforward though somewhat tedious. We therefore only sketch it in general lines. The isotropy of the hierarchy equations implies that pieces of $\boldsymbol{\mathcal{F}}_n,\boldsymbol{\mathcal{H}}_n$ with definite j,m, maintain their transformations properties under rotation *after* the linear and isotropic operations of the equation have been performed. For example, if $\mathcal{F}_n^{\alpha_1\ldots\alpha_n}(\mathbf{r}_0|\mathbf{X}_1,\ldots,\mathbf{X}_n)$ belongs to the irreducible representation (j,m), then so will the tensor fields:

$$
\partial^{(r_k)}_{\alpha_i}\mathcal{F}_n^{\alpha_1\ldots\alpha_n},\ \partial^2_{(r_k)}\mathcal{F}_n^{\alpha_1\ldots\alpha_n},\quad \text{etc...}
$$

although, they may belong to different $\mathcal{S}^n_p$ spaces (i.e., have one less or one more indices). Therefore, if we choose the bases $\left\{\mathbf{B}^{(n)}_{qjm}\right\}$ to be orthonormal, plug the expansion (16) into the hierarchy equations equations (13, 15), and take the inner product with $\mathbf{B}^{(n)}_{qjm}$, we will obtain new equations for the scalar functions $F^{(n)}_{qjm},H^{(n)}_{qjm}$:

$$
\begin{aligned}
&\partial_{t_1}F^{(n)}_{qjm}(r_0,X_1,\ldots,X_n) \qquad (17)\\
=&-\sum_{q'}\left\langle\left(\partial^{\alpha_1}_{(r_1)}+\partial^{\alpha_1}_{(r_1')}\right)H^{(n)}_{q'jm}(r_0,X_1,\ldots,X_n)\mathbf{B}^{(n-1)}_{q'jm},\mathbf{B}^{(n)}_{qjm}\right\rangle\\
&-\sum_{q'}\left\langle\partial^{(r_1)}_{\mu}F^{(n+1)}_{q'jm}(r_0,\tilde{X},X_1,\ldots,X_n)\mathbf{B}^{(n+1)}_{q'jm},\mathbf{B}^{(n)}_{qjm}\right\rangle\\
&-\sum_{q'}\left\langle\partial^{(r_1')}_{\mu}F^{(n+1)}_{q'jm}(r_0,\tilde{X}',X_1,\ldots,X_n)\mathbf{B}^{(n+1)}_{q'jm},\mathbf{B}^{(n)}_{qjm}\right\rangle\\
&+\nu\sum_{q'}\left\langle\left(\partial^2_{(r_1)}+\partial^2_{(r_1')}\right)F^{(n)}_{q'jm}(r_0,X_1,\ldots,X_n)\mathbf{B}^{(n)}_{q'jm},\mathbf{B}^{(n)}_{qjm}\right\rangle,
\end{aligned}
$$

$$
\begin{aligned}
&\sum_{q'}\left\langle\partial^{(r_1)}_{\alpha_1}F^{(n)}_{q'jm}(r_0,X_1,\ldots,X_n)\mathbf{B}^{(n)}_{q'jm},\mathbf{B}^{(n-1)}_{qjm}\right\rangle=0\ , \qquad (18)\\
&\sum_{q'}\left\langle\partial^{(r_1')}_{\alpha_1}F^{(n)}_{q'jm}(r_0,X_1,\ldots,X_n)\mathbf{B}^{(n)}_{q'jm},\mathbf{B}^{(n-1)}_{qjm}\right\rangle=0\ .
\end{aligned}
$$

Note that in the above equations, $\langle \cdot \rangle$ denote the inner-product in the $\mathcal{S}_p^n$ spaces. Also, the sums over q', j', m' from (16) was reduced to a sum over q' only - due to the isotropy. We thus see explicitly from (17,18) the decoupling of the equations for different j, m.

A. Rescaling Symmetry and Anomalous Exponents

The hierarchical equations simplify somewhat in the limit of infinite Reynolds number Re$\to \infty$. This limit is equivalent to $\nu \to 0$, in which the last term in Eqs.(17) can be neglected with impunity. It was pointed out before [3] that this is the main advantage of using "fully unfused" correlation functions in which all the coordinates are distinct: there is nothing to compensate for the vanishing of the viscosity in the $\nu \to 0$ limit. Once the viscous term is discarded, the rest of the equations . exhibit invariance to rescaling under the following rescaling group:

$$\begin{aligned} \mathbf{r}_i \to \lambda \mathbf{r}_i, \quad t_i \to \lambda^{1-h} t_1, \quad & F^{(n)}_{qjm} \to \lambda^{nh+\mathcal{Z}_j(h)} F^{(n)}_{qjm} \ , \\ & H^{(n)}_{qjm} \to \lambda^{(n+1)h+\mathcal{Z}_j(h)} H^{(n)}_{qjm} \ , \end{aligned} \tag{19}$$

as can be verified by direct substitution. In (19) λ and h are arbitrary scalars, and $\mathcal{Z}_j(h)$ is an arbitrary n-independent scalar function. We endow it with an index j since we expect, and see below, that $\mathcal{Z}_j(h)$ will differ in different j-sectors, but not in different m-sectors.

As a consequence of the rescaling symmetry we can seek solutions that do not mix values of h. We define $\tilde{F}^{(n)}_{qjm,h}$ and $\tilde{H}^{(n)}_{qjm,h}$ as the quantities that solve the equations of motion on an h-slice, which are the same as equations (17) without the viscous term. The important property of the solution on an h-slice is that it is a homogeneous function of all its arguments in the sense that

$$\begin{aligned} & \tilde{F}_{qjm,h}(\lambda r_0, \lambda r_1, \lambda r'_1, \lambda^{1-h} t_1, \dots \lambda r_n, \lambda r'_n, \lambda^{1-h} t_n) = \\ & \lambda^{nh+\mathcal{Z}_j(h)} \tilde{F}_{qjm,h}(r_0, r_1, r'_1, t_1, \dots r_n, r'_n, t_n) \ . \end{aligned} \tag{20}$$

It should be stressed that the quantity $F^{(n)}_{qjm}$ itself is *not* homogeneous in its arguments. It has been discovered in [8] and stressed in [3] that time-correlation functions in turbulence

do not satisfy dynamic scaling in the sense of Eq.(20). Indeed, the solution of Eq.(17) is a sum over contributions on h-slices,

$$F^{(n)}_{qjm}(r_0, X_1 \dots, X_n) = \int_{h_{\min}}^{h_{\max}} d\mu(h) \tilde{F}^{(n)}_{qjm,h}(r_0, X_1, \dots, X_n) \ , \tag{21}$$

with $\mu(h)$ some unknown measure that needs to be obtained from boundary conditions. Eq.(21) can be endowed with further meaning by rescaling coordinates and times according to

$$\rho_j \equiv \mathbf{r}_j/R_n, \ \rho'_j \equiv \mathbf{r}'_j/R_n, \ \tau_j \equiv t_j/t_{R_n,h} \ , \tag{22}$$

where R_n and $t_{R_n,h}$ are defined as the typical scale of separation of the set of coordinates and the typical times scale on that scale on an h -slice:

$$R_n^2 \equiv \frac{1}{n} \sum_{j=1}^{n} |\mathbf{r}_j - \mathbf{r}'_j|^2 \ , \tag{23}$$

$$t_{R_n,h} \equiv \frac{R}{U} \left(\frac{L}{R}\right)^h \ . \tag{24}$$

Here U is the typical velocity on the outer scale of turbulence L. Defining now

$$\Xi_j \equiv (\rho_j, \rho'_j, \tau_j) \ , \tag{25}$$

Eq.(21) can be written, using the rescaling property on an h slice as

$$\begin{aligned} &F^{(n)}_{qjm}(X_1, \dots, X_n) = \\ &U^n \int_{h_{\min}}^{h_{\max}} d\mu(h) \left(\frac{R_n}{L}\right)^{nh+\mathcal{Z}_j(h)} \tilde{F}^{(n)}_{qjm,h}(\Xi_1, \dots, \Xi_n) \end{aligned} \tag{26}$$

This form is known as the "multi-fractal" form [9–11]. The scaling exponents characterizing $a^{(n)}_{qjm}$ are obtained from a saddle-point calculation in the limit $R/L \to 0$ as $\min_h\{nh+\mathcal{Z}_j(h)\}$.

It was explained in [3] that $\mathcal{Z}_j(h)$ is obtained from a solvability condition of the hierarchy of equations (17). In particular the numerical value of the function $\mathcal{Z}_j(h)$, and consequently of the scaling exponents which are determined by the saddle point integral, depend on the *coefficients* in the equations (17). We found that the scalar functions associated with the different j -irreducible representations, $F^{(n)}_{qjm}(X_1, \dots, X_n)$, satisfy equations with different

coefficients, depending on inner products of the basis functions $\mathbf{B}_{qjm}$. Accordingly we expect the scalar function $\mathcal{Z}_j(h)$ to change from sector to sector. If the functions $F^{(n)}_{qjm}$ are characterized by anomalous exponents, they may be different for different j. On the other hand, for the same j the equations mix different m (and q) components, and unless there is an additional symmetry to SO(3), we do not expect different contributions with the same j to exhibit different exponents. In the next section we will demonstrate explicitly in the context of 3rd order correlation functions how the existence of an additional symmetry, in that case parity, brings about a foliation of a j sector into two sub-sectors which exhibit two different scaling exponents.

IV. EXAMPLE: KOLMOGOROV'S "FOUR-FIFTH LAW" AND THE FOLIATION TO DIFFERENT J'S

One of the best known results in the statistical theory of turbulence is Kolmogorov's "four-fifth law" which was discovered in 1941 [12]. This law pertains to the third order moment of longitudinal velocity differences $\delta u_l(\mathbf{r},\mathbf{R},t) \equiv [\mathbf{u}(\mathbf{r}+\mathbf{R},t) - \mathbf{u}(\mathbf{r},t)]\cdot\mathbf{R}/R$ where $\mathbf{u}(\mathbf{r},t)$ is the Eulerian velocity field of the turbulent fluid. The fourth-fifth law states that in homogeneous, isotropic and stationary turbulence, in the limit of vanishing kinematic viscosity $\nu \to 0$

$$\left\langle [\delta u_l(\mathbf{r},\mathbf{R},t)]^3 \right\rangle = -\tfrac{4}{5}\bar{\epsilon}R \ , \tag{27}$$

where $\bar{\epsilon}$ is the mean energy flux per unit time and mass $\bar{\epsilon} \equiv \nu \left\langle |\nabla_\alpha u_\beta|^2 \right\rangle$. The only assumption needed to derive this law is that the dissipation is finite in the limit $\nu \to 0$.

In this section we revisit this law by finding the full tensorial form of the $j=0$ component of the 3rd order correlation function. We *do not need* to assume isotropy of the turbulence at any stage of the development; the isotropy of the equations of motion suffices to decouple the $j=0$ contribution from all the rest, and in this case we have enough equations to determine the $j=0$ component of the tensor completely. We will also show that the $j=0$

component has two subcomponents with different scaling exponents. These subcomponents have different parity, and therefore are further decoupled in the equations of motion. The usual fourth-fifth law pertains to the components that have odd parity. One can derive an additional exact relation that pertains to the even parity components [13].

Defining the velocity $\mathbf{v}(\mathbf{r},t)$ as $\mathbf{v}(\mathbf{r},t) \equiv \mathbf{u}(\mathbf{r},t) - \langle \mathbf{u} \rangle$ we consider the simultaneous 3rd order tensor correlation function which depends on two space points:

$$J^{\alpha,\beta\gamma}(\mathbf{R}) \equiv \left\langle v^\alpha(\mathbf{r}+\mathbf{R},t) v^\beta(\mathbf{r},t) v^\gamma(\mathbf{r},t) \right\rangle \ . \tag{28}$$

We show [13] that in the limit $\nu \to 0$, under the same assumption leading to the fourth-fifth law, this correlation function reads

$$J^{\alpha,\beta\gamma}(\mathbf{R}) = - \frac{\bar{\epsilon}}{10}(R^\gamma \delta^{\alpha\beta} + R^\beta \delta^{\alpha\gamma} - \frac{2}{3} R^\alpha \delta^{\beta\gamma}) \tag{29}$$

$$- \frac{\bar{h}}{30}(\epsilon^{\alpha\beta\delta} R^\gamma + \epsilon^{\alpha\gamma\delta} R^\beta) R_\delta \ . \tag{30}$$

The quantity $\bar{h}$ is the mean dissipation of helicity per unit mass and time

$$\bar{h} \equiv \nu \left\langle (\partial^\alpha u^\beta)(\partial^\alpha [\nabla \times \mathbf{u}]^\beta) \right\rangle \ , \tag{31}$$

The new result (30) (derived firstly in [13] and [14]) can be also displayed in a form that depends on $\bar{h}$ alone by introducing the longitudinal and transverse parts of $\mathbf{u}$: the longitudinal part is $\mathbf{u}_l \equiv \mathbf{R}(\mathbf{u} \cdot \mathbf{R})/R^2$ and the transverse part is $\mathbf{u}_t \equiv \mathbf{u} - \mathbf{u}_l$. In addition we have $\delta \mathbf{u}_l(\mathbf{r},\mathbf{R},t) \equiv \delta u_l(\mathbf{r},\mathbf{R},t)\mathbf{R}/R$. In terms of these quantities we can propose a "two fifteenth law" that pertains to the $j=0$ component of the following correlation function:

$$\langle [\delta \mathbf{u}_l(\mathbf{r},\mathbf{R},t)] \cdot [\mathbf{u}_t(\mathbf{R}+\mathbf{r},t) \times \mathbf{u}_t(\mathbf{r},t)] \rangle = \tfrac{2}{15} \bar{h} R^2 \ . \tag{32}$$

We note that this result holds also when we replace $\mathbf{u}$ by $\mathbf{v}$ everywhere.

To derive the result (30) we start from the correlation function $J^{\alpha,\beta\gamma}(\mathbf{R})$ which is symmetric with respect to exchange of the indices β and γ as is clear from the definition. Using the symmetry the most general form of the $j=0$ component of this tensor can be written by observation (Sect. 2b):

$$J^{\alpha,\beta\gamma}(\mathbf{R}) = a_1(R)[\delta^{\alpha\beta}\hat{R}^\gamma + \delta^{\alpha\gamma}\hat{R}^\beta + \delta^{\beta\gamma}\hat{R}^\alpha] \tag{33}$$

$$+a_2(R)[\delta^{\alpha\beta}\hat{R}^\gamma + \delta^{\alpha\gamma}\hat{R}^\beta - 2\delta^{\beta\gamma}\hat{R}^\alpha]$$

$$+a_3(R)[\delta^{\alpha\beta}\hat{R}^\gamma + \delta^{\alpha\gamma}\hat{R}^\beta + \delta^{\beta\gamma}\hat{R}^\alpha - 5R^\alpha\hat{R}^\beta\hat{R}^\gamma/R^2]$$

$$+a_4(R)[\epsilon^{\alpha\beta\delta}\hat{R}^\gamma + \epsilon^{\alpha\gamma\delta}\hat{R}^\beta]\hat{R}_\delta \tag{34}$$

This form is precisely of the type $\sum_q a_{qjm}\mathbf{B}_{qjm}$ for the isotropic sector $j = m = 0$.

Not all the coefficients are independent for incompressible flows. Requiring $\partial J^{\alpha,\beta\gamma}(\mathbf{R})/\partial R^\alpha = 0$ leads to two relations among the coefficients:

$$\begin{aligned} &\Big(\frac{d}{dR} + \frac{4}{R}\Big)a_3(R) = \frac{2}{3}[\frac{d}{dR} - \ \frac{1}{R}][a_1(R) + a_2(R)] \ , \\ &\Big(\frac{d}{dR} + \frac{2}{R}\Big)[5a_1(R) - 4a_2(R)] = 0 \ . \end{aligned} \tag{35}$$

As we have two conditions relating the three coefficients $a_1,\ a_2,\ a_3$ only one of them is independent. Kolmogorov's derivation [12] related the rate of energy dissipation to the value of the remaining unknown. Here the coefficient a_4 remains undetermined by the incompressibility constraint. It belongs to a component of odd parity; since the equations of motion and the incompressibility constraint are invariant under parity transformation it decouples altogether and needs to be determined separately.

Kolmogorov's derivation can be paraphrased in a simple manner. Begin with the second order structure function which is related to the energy of R -scale motions

$$S_2(R) \equiv \left\langle |\mathbf{u}(\mathbf{R}+\mathbf{r},t) - \mathbf{u}(\mathbf{r},t)|^2 \right\rangle \ . \tag{36}$$

Computing the rate of change of this (time-independent) function from the Navier-Stokes equations we find

$$0 = \frac{\partial S_2(R)}{2\partial t} = -\mathcal{D}_2(R) - 2\bar{\epsilon} + \nu\nabla^2 S_2(R) \ , \tag{37}$$

where $\mathcal{D}_2(R)$ stems from the nonlinear term $(\mathbf{u}\cdot\nabla)\mathbf{u}$ and as a result it consists of a correlation function including a velocity derivative. The conservation of energy allows the derivative to be taken outside the correlation function:

$$\mathcal{D}_2(R) \equiv \frac{\partial}{\partial R^\beta}\langle u^\alpha(\mathbf{r},t)u^\alpha(\mathbf{r}+\mathbf{R},t)[u^\beta(\mathbf{r},t) \\ -u^\beta(\mathbf{r}+\mathbf{R},t)]\rangle \ . \tag{38}$$

In terms of the function of Eq. (28) we can write

$$\mathcal{D}_2(R) = \frac{\partial}{\partial R^\beta}\Big[J^{\alpha,\beta\alpha}(\mathbf{R},t) - J^{\alpha,\beta\alpha}(-\mathbf{R},t)\Big] \ . \tag{39}$$

Note that Eq. (28) is written in terms of $\mathbf{v}$ rather than $\mathbf{u}$, but using the incompressibility constraint we can easily prove that Eq. (38) can also be identically written in terms of $\mathbf{v}$ rather than $\mathbf{u}$. We proceed using Eq. (33) in Eq. (39), and find

$$\mathcal{D}_2(R) = 2\frac{\partial}{\partial R^\beta}\hat{R}^\beta[5a_1(R) + 2\tilde{a}_1(R)] \ . \tag{40}$$

For R in the inertial interval, and for $\nu \to 0$, we can read from Eq. (37) $\mathcal{D}_2(R) = -2\bar{\epsilon}$ and therefore have the third relation that is needed to solve all the three unknown coefficients. A calculation leads to

$$a_1(R) = -2\bar{\epsilon}R/45 \ , \quad a_2 = -\bar{\epsilon}R/18 \ , \quad a_3 = 0 \ . \tag{41}$$

The choice of the structure function $S_2(R)$ leads to Eq.(39) in which the odd parity components disappear, leaving $a_4(R)$ undetermined. Another correlation function is needed in order to remedy the situation. Since the helicity is $\mathbf{u}\cdot[\nabla\times\mathbf{u}]$, we seek a correlation function which is related to the helicity of eddies of scale of R:

$$T_2(R) \equiv \langle[\mathbf{u}(\mathbf{R}+\mathbf{r},t) - \mathbf{u}(\mathbf{r},t)] \\ \cdot[\nabla\times\mathbf{u}(\mathbf{r}+\mathbf{R},t) - \nabla\times\mathbf{u}(\mathbf{r},t)]\rangle \ . \tag{42}$$

The proper choice of this correlation function is the crucial idea here. The rest is a straightforward calculation. Using the Navier-Stokes equations to compute the rate of change of this quantity we find

$$0 = \frac{\partial T_2(R)}{2\partial t} = -G_2(R) - 2\bar{h} - \nu\nabla^2 T_2(R) \ , \tag{43}$$

which is the analog of (37), and where

$$G_2(R) = \{\langle \mathbf{u}(\mathbf{r},t) \cdot [\nabla_R \times [\mathbf{u}(\mathbf{r}+\mathbf{R},t) \times [\nabla_R \times \mathbf{u}(\mathbf{r}+\mathbf{R},t)]]]\rangle\} + \{\text{term } \mathbf{R} \to -\mathbf{R}\} \ . \tag{44}$$

The conservation of helicity allows the extraction of two derivatives outside the correlation functions. The result can be expressed in terms of our definition (28):

$$G_2(R) = \frac{\partial}{\partial R^\lambda}\frac{\partial}{\partial R^\kappa}\epsilon_{\alpha\lambda\mu}\epsilon_{\mu\beta\nu}\epsilon_{\nu\kappa\gamma}[J^{\alpha,\beta\gamma}(\mathbf{R}) + J^{\alpha,\beta\gamma}(-\mathbf{R})] \ . \tag{45}$$

Substituting Eq. (33) we find

$$G_2(R) = 2\frac{\partial^2}{\partial R^\lambda \partial R^\kappa} b_2(R)[\hat{R}^\lambda \hat{R}^\kappa - \delta^{\lambda\kappa}] \ , \tag{46}$$

which is the analog of Eq. (40). Using Eq. (43) in the inertial interval in the limit $\nu \to 0$ we find the differential equation

$$\frac{d^2 a_4(R)}{dR^2} + 5\frac{1}{R}\frac{db_2(R)}{dR} + \frac{3a_4(R)}{R^2} = -\frac{\bar{h}}{2} \ . \tag{47}$$

The general solution of this equation is

$$a_4(R) = -\bar{h}R^2/30 + \alpha_1 R^{-3} + \alpha_2 R^{-1} \ . \tag{48}$$

Requiring finite solutions in the limit $R \to 0$ means that $\alpha_1 = \alpha_2 = 0$. Accordingly we end up with Eq. (30). We restate again, that in the preceding derivation, we did not assume that our turbulence were isotropic. Other terms of $J^{\alpha,\beta\gamma}(\mathbf{R})$ with $j \neq 0$ can possibly contribute to the total $J^{\alpha,\beta\gamma}(\mathbf{R})$. However, under the assumptions of homogeneity and finite energy and helicity dissipations, the R dependence of the isotropic part of $J^{\alpha,\beta\gamma}(\mathbf{R})$ is necessarily as stated in (30).

It should be noted, that a parallel calculation can be easily carried out for the $j > 0$ sectors of the third order correlation function $J^{\alpha,\beta\gamma}(\mathbf{R})$. In these sectors however, there are more irreducible representations than there are in the $j = 0$ sector (to be exact, there is a total of 18 representations for each $j > 2$. 10 of them with $(-)^{j+1}$ parity and 8 of them with

$(-)^j$ parity). As a result, for $j > 0$ we get more unknown functions $a_{qjm}(R)$ than equations, and hence we cannot obtain a full solution. Our failure in obtaining a complete set of equations for $j > 0$ sectors, can be attributed to the inadequacy of a turbulence theory that involves only moments which are simultaneous in time. The $j = 0$ sector is indeed unique in the sense that its low number of irreducible representations makes a full solution possible.

V. EXAMPLE: ANALYSIS OF ANISOTROPY IN ATMOSPHERIC TURBULENCE

In this section we present experimental evidence for the utility of the expansion (8) in terms of the irreducible representations. The analysis of the experimental data was done in collaboration with B. Dhruva, S. Kurien and K.R. Sreenivasan, and the reader is referred to the details of [15]. In that work we focused on the 2nd rank tensor structure functions of velocity differences

$$S^{\alpha\beta}(\mathbf{R}) \equiv \left\langle [u^\alpha(\mathbf{r}+\mathbf{R}) - u^\alpha(\mathbf{r})] \left[u^\beta(\mathbf{r}+\mathbf{R}) - u^\beta(\mathbf{r})\right] \right\rangle \ . \tag{49}$$

where *homogeneity* of the flow is assumed, but not isotropy. This object is symmetric in its indices and has even parity in $\mathbf{R}$. In addition it is expected to scale with $\mathbf{R}$ in the inertial range, with possibly different scaling exponent characterizing contributions of different j. We demonstrated in [15] that one can usefully represent $S^{\alpha\beta}(\mathbf{R})$ in the form:

$$S^{\alpha\beta}(\mathbf{R}) = \sum_{qjm} a_{qjm} |\mathbf{R}|^{\zeta_2^{(j)}} B^{\alpha\beta}_{qjm}(\hat{\mathbf{R}}) \tag{50}$$

where a_{qjm} are some numerical coefficients, $B^{\alpha\beta}_{qjm}(\hat{\mathbf{R}})$ are the tensor basis of $\mathcal{S}_1^2$ with a definite j, m, and $\zeta_2^{(j)}$ are the exponents associated with the j's irreducible representation. The isotropic exponent, $\zeta_2^{(0)}$, will be referred to shortly as ζ_2. We note that the coefficients a_{qjm} are *not* arbitrary numerical coefficients, because of the constraints imposed by the incompressibility of the flow. In Appendix A we derive the explicit form of $B^{\alpha\beta}_{qjm}(\hat{\mathbf{R}})$ and the necessary relations among the a_{qjm}'s. The theoretical development of Appendix A serves

as a basis for the data analysis; we leave it in the appendix since it is somewhat lengthy. Nevertheless the interested reader may find it useful for situations that differ from the one treated below.

The data that we want to consider were taken at Taylor microscale Reynolds numbers of about 10,000 [15]. The data were acquired simultaneously from two single-wire probes separated by $\Delta = 55$ cm nominally orthogonal to the mean wind direction. The two probes were mounted at a height of 6 m over a flat desert with a long fetch. The Kolmogorov scale was about 0.75 cm. Details of the experimental setup can be found in ref. [15]. In that reference one can find details of another data set that was analyzed in the same fashion, leading to results in agreement with those reviewed here.

Firstly we tested the isotropy of the flow for separations of the order of Δ. Using the standard Taylor hypothesis, define the "transverse" structure function across Δ as $S_T(\Delta) \equiv \langle [u_1(\bar{U}t) - u_2(\bar{U}t)]^2\rangle$ and the "longitudinal" structure function as $S_L(\Delta) \equiv \langle [u_1(\bar{U}t + \bar{U}t_\Delta) - u_1(\bar{U}t)]^2\rangle$ where $t_\Delta = \Delta/\bar{U}$. If the flow were isotropic we would expect [16]

$$S_T(\Delta) = S_L(\Delta) + \frac{\Delta}{2}\frac{\partial S_L(\Delta)}{\partial \Delta} \ . \tag{51}$$

In the isotropic state both components scale with the same exponent, $S_{T,L}(\Delta) \propto \Delta^{\zeta_2}$, and their ratio is computed from (51) to be $1 + \zeta_2/2 \approx 1.35$ where $\zeta_2 \approx 0.69$ (see below). The experimental ratio was found to be 1.32, indicating that the anisotropy at the scale Δ is small. We expect that the effects of anisotropy should be most pronounced on the larger scales.

Next, we found the functional form of the basis tensors $B^{\alpha\beta}_{qjm}(\hat{\mathbf{R}})$ and the algebraic relations among the coefficients a_{qjm} according to the discussion in the last paragraph of Appendix A.

Since the anisotropies are not huge, we focused on the lowest order corrections to the isotropic ($j = 0$) contribution. In other words, we wrote

$$S^{\alpha\beta}(\mathbf{R}) = S^{\alpha\beta}_{j=0}(\mathbf{R}) + S^{\alpha\beta}_{j=1}(\mathbf{R}) + S^{\alpha\beta}_{j=2}(\mathbf{R}) + S^{\alpha\beta}_{j=3}(\mathbf{R}) \ . \tag{52}$$

We defined the coordinate system such that the mean wind direction was along the 3-axis, and the separation between the two probes was along the 1-axis. By *assuming* axial symmetry along the mean wind direction, the tensors $S^{\alpha\beta}_{j=0}(\mathbf{R}),\ S^{\alpha\beta}_{j=1}(\mathbf{R}),\ S^{\alpha\beta}_{j=2}(\mathbf{R})$ were to contain *only* the $m=0$ components. In addition, since the two probes measured the velocity field only in the mean wind direction, we had only the values of $S^{33}(\mathbf{R})$ in the $1-3$ plane. In such a case, it turns out that only the even j's have a non-vanishing contribution. We therefore used the trial tensor:

$$S^{\alpha\beta}(\mathbf{R}) = S^{\alpha\beta}_{j=0}(\mathbf{R}) + S^{\alpha\beta}_{j=2}(\mathbf{R}) \ ,$$

$$\begin{aligned} S^{\alpha\beta}_{j=0}(\mathbf{R}) &= c_0 \left(\frac{R}{\Delta}\right)^{\zeta_2} \left[(2+\zeta_2)\delta^{\alpha\beta} - \zeta_2 \frac{R^\alpha R^\beta}{R^2}\right] \ , \\ S^{\alpha\beta}_{j=2}(\mathbf{R}) &= a S^{\alpha\beta}_{j=2,q=1}(\mathbf{R}) + b S^{\alpha\beta}_{j=2,q=2}(\mathbf{R}). \end{aligned}$$

Where $S^{\alpha\beta}_{j=2,q=1}(\mathbf{R}),\ S^{\alpha\beta}_{j=2,q=2}(\mathbf{R})$ are given by:

$$\begin{aligned} S^{\alpha\beta}_{j=2,q=1}(\mathbf{R}) = \left(\frac{R}{\Delta}\right)^{\zeta_2^{(2)}} & \Big[(\zeta_2^{(2)}-2)\delta^{\alpha\beta} - \zeta_2^{(2)}(\zeta_2^{(2)}+6)\delta^{\alpha\beta}\frac{(\boldsymbol{k}\cdot\boldsymbol{R})^2}{R^2} + 2\zeta_2^{(2)}(\zeta_2^{(2)}-2)\frac{R^\alpha R^\beta(\boldsymbol{k}\cdot\boldsymbol{R})^2}{R^4} \\ & + ([\zeta_2^{(2)}]^2 + 3\zeta_2^{(2)} + 6)k^\alpha k^\beta - \frac{\zeta_2^{(2)}(\zeta_2^{(2)}-2)}{R^2}(R^\alpha k^\beta + R^\beta k^\alpha)(\boldsymbol{k}\cdot\boldsymbol{R})\Big] \ , \qquad (53) \\ S^{\alpha\beta}_{j=2,q=2}(\mathbf{R}) = \left(\frac{R}{\Delta}\right)^{\zeta_2^{(2)}} & \Big[-(\zeta_2^{(2)}+3)(\zeta_2^{(2)}+2)\delta^{\alpha\beta}(\boldsymbol{k}\cdot\boldsymbol{R})^2 + (\zeta_2^{(2)}-2)\frac{R^\alpha R^\beta}{R^2} \\ & + (\zeta_2^{(2)}+3)(\zeta_2^{(2)}+2)k^\alpha k^\beta + (2\zeta_2^{(2)}+1)(\zeta_2^{(2)}-2) \\ & \times \frac{R^\alpha R^\beta(\boldsymbol{k}\cdot\boldsymbol{R})^2}{R^4} - ([\zeta_2^{(2)}]^2-4)(R^\alpha k^\beta + R^\beta k^\alpha)(\boldsymbol{k}\cdot\boldsymbol{R})\Big] \ . \end{aligned}$$

The vector $\boldsymbol{k}$ was taken to be along the mean wind direction. ζ_2 is the isotropic exponent, while $\zeta_2^{(2)}$ is the $j=2$ exponent. $c_0,\ a,\ b$ are the non-universal weights of the components of $S^{\alpha\beta}(\mathbf{R})$. In order to reduce the number of unknown quantities, the exponent of the isotropic part of $S^{\alpha\beta}(\mathbf{R})$ was assumed to be known: $\zeta_2 = 0.69$. The values of $c_0,\ a,\ b,\ \zeta_2^{(2)}$ were to be found from the experimental data.

Using spherical coordinates, the trial tensor S^{33} in the $1-3$ plane took the following form:

$$
\begin{aligned}
S^{33}(R,\theta,\phi=0) &= S^{33}_{j=0}(R,\theta,\phi=0)+S^{33}_{j=2}(R,\theta,\phi=0) \\
&= c_0\left(\frac{R}{\Delta}\right)^{0.69}\Big[2+0.69-0.69\cos^2\theta\Big] \\
&\quad +a\left(\frac{R}{\Delta}\right)^{\zeta_2^{(2)}}\Big[(\zeta_2^{(2)}+2)^2-\zeta_2^{(2)}(3\zeta_2^{(2)}+2)\cos^2\theta \\
&\quad +2\zeta_2^{(2)}(\zeta_2^{(2)}-2)\cos^4\theta\Big] \\
&\quad +b\left(\frac{R}{\Delta}\right)^{\zeta_2^{(2)}}\Big[(\zeta_2^{(2)}+2)(\zeta_2^{(2)}+3)-\zeta_2^{(2)}(3\zeta_2^{(2)}+4)\cos^2\theta \\
&\quad +(2\zeta_2^{(2)}+1)(\zeta_2^{(2)}-2)\cos^4\theta\Big].
\end{aligned}
\tag{54}
$$

where θ is the angle between $\mathbf{k}$ and $\mathbf{R}$. The fitting of the trial tensor to the data was done along two paths in the (R,θ) space:

- $\theta=0$ A single probe measurement.
- $R\sin\theta=\Delta$ Two probes measurement.

Fig. 1 shows the best fit to the data. For each type of data, two fits were performed: A fit of the isotropic part only (panel a), and a fit of both isotropic and $j=2$ components (panel b). The excellent fits in panel (b) is a good support for the present mode of analysis.

In Ref. [15] Fig.3 we showed the determination of $\zeta_2^{(2)}$ from a least-square fit. The optimal value of this exponent and the uncertainty determined from that plot is $\zeta_2^{(2)}=1.38\pm 0.15$. It should be understood that the exponent $\zeta_2^{(2)}$ (and also $\zeta_2^{(1)}$ that is unavailable from the present measurements) are just the smallest exponents in the hierarchy $\zeta_2^{(j)}$ that characterizes higher order irreducible representations indexed by j. The study of these exponents has only begun here, and considerable experimental and theoretical effort is needed to reach firm conclusions regarding their universality and numerical values. We expect the exponents to be a non-decreasing function of j, explaining why the highest values of j are being peeled off quickly when R decreases. Nevertheless, the lower order values of $\zeta_2^{(j)}$ can be measured and computed. In Ref. [15] we presented an additional set of experimental data, and demonstrated that the numerical value of $\zeta_2^{(2)}$ appears universal.

VI. CONCLUDING REMARKS

The aim of this paper has been to introduce the exploration of the scaling properties of turbulent statistics in the anisotropic sectors. The main novel theoretical development is described in Sect.2. It is explained there that the linearity of the equations for the fully unfused correlation functions together with the invariance to rotations, foliates the solutions into sectors characterized by the j, m designations of the irreducible representations of the SO(3) symmetry group. As a consequence we expect the different sectors to be characterized by different (anomalous) scaling exponents. This observation opens up a new and interesting research arena for theory and experiments.

In Sect.3 we presented a derivation of the form of the second order structure function in the higher j-sectors, and used a recent experimental data analysis as an example of the utility and importance of the present approach. The main result of this section, besides the theoretical forms that can be used for further data analysis, is that the scaling range in turbulence is much larger than expected. One just needs to acknowledge the existence of anisotropic contributions to obtain scaling ranges that go all the way from the Kolmogorov scale to the outer scale of turbulence.

It is our belief that this paper does not exhaust the issue of anisotropic contributions to turbulent statistics. It is only the beginning of a rich research program that should be carried simultaneously by experimentalist and theorists.

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Chaos and Complexity.

APPENDIX A: THE GENERAL FORM OF THE 2ND RANK TENSOR

In this appendix we discuss the general structure of the second rank correlation functions

$$F^{\alpha\beta}(\mathbf{R}) \equiv \langle u^{\alpha}(\mathbf{r}+\mathbf{R})u^{\beta}(\mathbf{r})\rangle \ , \tag{A1}$$

In (A1) *homogeneity* of the flow is assumed, but not isotropy. Note that this object is more general than the structure function $S^{\alpha\beta}$ in being nonsymmetric in the indices, and having no definite parity. In light of the discussion in Sect. 2, when we expand this objects in terms of tensors with definite j, m, we expect each component to have a distinct dependence on the amplitude $R \equiv |\mathbf{R}|$. Accordingly, we wish to find the basis functions $B^{\alpha\beta}_{qjm}(\hat{\mathbf{R}})$, with which we can represent $F^{\alpha\beta}(\mathbf{R})$ in the form:

$$F^{\alpha\beta}(\mathbf{R}) = \sum_{qjm} a_{qjm}(R) B^{\alpha\beta}_{qjm}(\hat{\mathbf{R}}) \tag{A2}$$

and derive some constraints among the functions $a_{qjm}(R)$ that result from incompressibility. We shall see, that due to the isotropy of the incompressibility conditions, the constraints are among $a_{qjm}(R)$ with the *same* j, m only.

We begin by analyzing the incompressibility condition: An incompressible flow with constant density is characterized by the relation:

$$\partial_{\alpha} u^{\alpha}(\mathbf{r}, t) = 0$$

as a result, one immediately gets the following constraints on $F^{\alpha\beta}(\mathbf{R})$:

$$\partial_{\alpha} F^{\alpha\beta}(\mathbf{R}) = 0$$
$$\partial_{\beta} F^{\alpha\beta}(\mathbf{R}) = 0.$$

Plugging the trial tensor (A2) into the last two equations we obtain 2 equations connecting the different a_{qjm}:

$$\partial_\alpha \sum_{qjm} a_{qjm}(R) B^{\alpha\beta}_{qjm}(\hat{\mathbf{R}}) = 0 \tag{A3}$$
$$\partial_\beta \sum_{qjm} a_{qjm}(R) B^{\alpha\beta}_{qjm}(\hat{\mathbf{R}}) = 0$$

We first notice that the differentiation action is isotropic. As a result, if $T^{\alpha\beta}(\mathbf{R})$ is some arbitrary tensor with a definite j, m transformation properties, then the tensor $\partial_\alpha T^{\alpha\beta}(\mathbf{R})$ will have *the same* j, m transformation properties. Components with different j, m are linearly independent. Therefore equations (A3) should hold for each j, m separately.

Next, we observe that (A3) are invariant under the transformation $F^{\alpha\beta} \longrightarrow F^{\beta\alpha}$. As a result, the symmetric and anti-symmetric parts of $F^{\alpha\beta}$ should satisfy (A3) independently. To see that, let us write $F^{\alpha\beta}$ as a sum of a symmetric term and an anti-symmetric term: $F^{\alpha\beta} = F^{\alpha\beta}_S + F^{\alpha\beta}_A$, we then get:

$$\partial_\alpha F^{\alpha\beta} = \partial_\alpha F^{\alpha\beta}_S + \partial_\alpha F^{\alpha\beta}_A = \partial_\alpha F^{\beta\alpha}_S - \partial_\alpha F^{\beta\alpha}_A = 0$$
$$\partial_\beta F^{\alpha\beta} = \partial_\beta F^{\alpha\beta}_S + \partial_\beta F^{\alpha\beta}_A = 0$$

from which we conclude:

$$\partial_\alpha F^{\alpha\beta}_S = \partial_\alpha F^{\alpha\beta}_A = 0$$

Finally, (A3) is invariant under the transformation $F^{\alpha\beta}(\mathbf{R}) \longrightarrow F^{\alpha\beta}(-\mathbf{R})$ and as a result the odd parity and the even parity parts of $F^{\alpha\beta}$ should fulfill (A3) independently. We conclude that a necessary and sufficient condition for (A3) to hold is that it holds separately for parts with definite j, m, definite symmetry in the α, β indices and a definite parity in $\mathbf{R}$:

$$\partial_\alpha \sum_q a_{qjm}(|\mathbf{R}|) B^{\alpha\beta}_{qjm}(\hat{\mathbf{R}}) = 0 \qquad \begin{array}{l}\text{summation is over} \\ B^{\alpha\beta}_{qjm} \text{ with definite symmetries}\end{array}$$

where the summation is over q such that $B^{\alpha\beta}_{qjm}$ has a definite indices symmetry and a definite parity.

According to (10) we can write these $B^{\alpha\beta}_{qjm}$ as:

1. $(-)^j$ parity, symmetric tensors:

- $B^{\alpha\beta}_{1,jm}(\hat{\mathbf{R}}) \equiv R^{-j}\delta^{\alpha\beta}\Phi_{jm}(R)$,
- $B^{\alpha\beta}_{7,jm}(\hat{\mathbf{R}}) \equiv R^{-j}\left[R^\alpha\partial^\beta + R^\beta\partial^\alpha\right]\Phi_{jm}(R\ )$,
- $B^{\alpha\beta}_{9,jm}(\hat{\mathbf{R}}) \equiv R^{-j-2}R^\alpha R^\beta\Phi_{jm}(R)$,
- $B^{\alpha\beta}_{5,jm}(\hat{\mathbf{R}}) \equiv R^{-j+2}\partial^\alpha\partial^\beta\Phi_{jm}(R)$.

2. $(-)^j$ parity, anti-symmetric tensors:

- $B^{\alpha\beta}_{3,jm}(\hat{\mathbf{R}}) \equiv R^{-j}\left[R^\alpha\partial^\beta - R^\beta\partial^\alpha\right]\Phi_{jm}(R\ )$.

3. $(-)^{j+1}$ parity, symmetric tensors:

- $B^{\alpha\beta}_{8,jm}(\hat{\mathbf{R}}) \equiv R^{-j-1}\left[R^\alpha\epsilon^{\beta\mu\nu}R_\mu\partial_\nu + R^\beta\epsilon^{\alpha\mu\nu}R_\mu\partial_\nu\right]\Phi_{jm}(R)$,
- $B^{\alpha\beta}_{6,jm}(\hat{\mathbf{R}}) \equiv R^{-j+1}\left[\epsilon^{\beta\mu\nu}R_\mu\partial_\nu\partial^\alpha + \epsilon^{\alpha\mu\nu}R_\mu\partial_\nu\partial^\beta\right]\Phi_{jm}(R)$.

4. $(-)^{j+1}$ parity, anti-symmetric tensors:

- $B^{\alpha\beta}_{4,jm}(\hat{\mathbf{R}}) \equiv R^{-j-1}\epsilon^{\alpha\beta\mu}R_\mu\Phi_{jm}(R)$,
- $B^{\alpha\beta}_{2,jm}(\hat{\mathbf{R}}) \equiv R^{-j+1}\epsilon^{\alpha\beta\mu}\partial_\mu\Phi_{jm}(R)$.

In order to differentiate these expressions we can use the following identities:

$$R^\alpha\partial_\alpha R^\zeta Y_{jm}(\hat{\mathbf{R}}) = \zeta R^\zeta Y_{jm}(\hat{\mathbf{R}})\ ,$$
$$\partial^\alpha\partial_\alpha R^\zeta Y_{jm}(\hat{\mathbf{R}}) = \left[\zeta\left(\zeta+1\right) - j(j+1)\right]R^{\zeta-2}Y_{jm}(\hat{x})$$

which give rise to:

$$R^\alpha\partial_\alpha\Phi_{jm}(\mathbf{R}) = j\Phi_{jm}(\mathbf{R})\ .$$
$$\partial^\alpha\partial_\alpha\Phi_{jm}(\mathbf{R}) = 0\ .$$

From this point, it is a matter of simple (though somewhat lengthy) algebra to derive the differential constraints among $a_{qjm}(R)$. The results are as follows:

1. $q \in \{1, 7, 9, 5\}$

$$a'_{1,jm}(R) - jR^{-1}a_{1,jm} + ja'_{7,jm} - j^2R^{-1}a_{7,jm} + a'_{9,jm} + 2R^{-1}a_{9,jm} = 0 \ , \tag{A4}$$

$$R^{-1}a_{1,jm} + a'_{7,jm} + 3R^{-1}a_{7,jm} + (j-1)\, a'_{5,jm} - \left(j^2 - 3j + 2\right) R^{-1}a_{5,jm} = 0 \ .$$

2. $q \in \{3\}$

$$a'_{3,jm} - jR^{-1}a_{3,jm} = 0 \ , \tag{A5}$$

$$a'_{3,jm} + R^{-1}a_{3,jm} = 0 \ .$$

Notice: These equations have no solutions other than: $a_{3,jm}(R) = 0$.

3. $q \in \{8, 6\}$

$$a'_{8,jm} + 3R^{-1}a_{8,jm} + (j-1)a'_{6,jm} - \left(j^2 - 2j + 1\right) R^{-1}a_{6,jm} = 0 \ . \tag{A6}$$

4. $q \in \{4, 2\}$

$$R^{-1}a_{4,jm} - a'_{2,jm} + (j-1)R^{-1}a_{2,jm} = 0 \ . \tag{A7}$$

There are obviously more unknowns than equations, since we merely exploited the incompressibility conditions. Nevertheless, we believe that the missing equations that arise from the dynamical hierarchy of equations will preserve the distinction between a_{qjm} of different j, m (again, due to the isotropy of these equations).

Note also, that the above analysis holds also for the second-order structure function

$$S^{\alpha\beta}(\mathbf{R}) \equiv \left\langle [u^\alpha(\mathbf{r}+\mathbf{R}) - u^\alpha(\mathbf{r})] \left[u^\beta(\mathbf{r}+\mathbf{R}) - u^\beta(\mathbf{r})\right] \right\rangle .$$

Only that in this case we should only consider the representations $q = 1, 7, 9, 5$ for even j and the representations $q = 8, 6$ for odd j. This follows from the fact that $S^{\alpha\beta}(\mathbf{R})$ is symmetric with respect to its indices and it has an even parity in $\mathbf{R}$. Also, in that case, it is possible

to go one step further by assuming a specific functional form for the $a_{q,jm}(R)$. We know that the $S^{\alpha\beta}(\mathbf{R})$ is expected scale in the inertial range, and we therefore may *assume*:

$$a_{q,jm}(R) \equiv c_{q,jm} R^{\zeta_2^{(j)}} .$$

where $c_{q,jm}$ are just numerical constants. If we now substitute this definition into the equations (A4,A6), we get a set of linear equations among the $c_{q,jm}$. These relations can be easily solved and give us two possible tensors for even j $(q = 1, 7, 9, 5)$ and one tensor form for odd j (from $q = 8, 6$). This kind of approach was taken in the two-probes experiment which is described in Sect. V.

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FIGURES

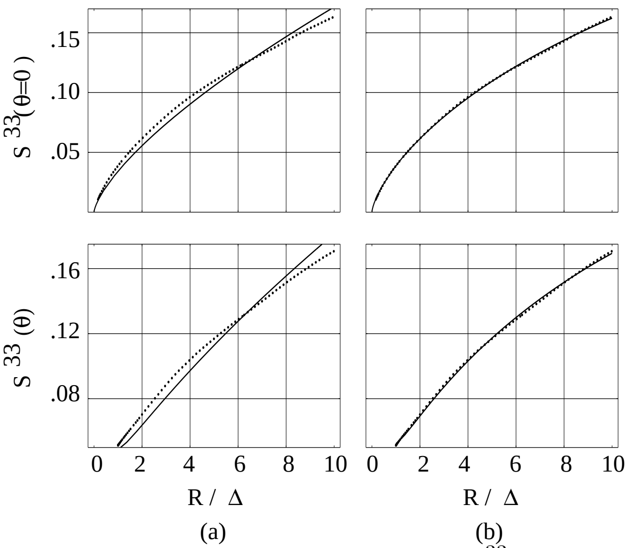

FIG. 1. The structure functions S^{33} for $\theta = 0$ and for non-zero θ computed for set I. The dots are for experimental data and the line is the analytic fit. Panel (a) presents fits to the $j = 0$ component only, and panel (b) to components $j = 0$ and $j = 2$ together.